USE OF THE BAYESIAN STATISTICS AND THE PRODUCT OF PROBABILITIES IN THE IONIZING RADIATION FIELD

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Abstract— **Nowadays, the probability of the intersection (PI) of two or more stochastic events or processes is calculated as the product of probabilities (PPs). The Bayes' theorem (BT) is widely used in the ionizing radiation field. We will show the PI is not only obtained as the PPs; but the minimum of their probabilities; and demonstrate that terms P(B|A) and P(A|B) in the BT are not new probabilistic metrics, but the own respective probabilities of B and A events. Mathematical derivations based on strong probabilistic foundations, and with their respective illustrations were our methodology. There are demonstrations of: 1) The two ways for determining the PI; and 2) Incoherencies of the BT. The tumor control probability (TCP) and normal tissue non-complication probability (NTCP0) of the radiation oncology treatments to patients with more than one target, calculated respectively as the product of TCP, and NTCP0 of each treatment, are excellent-practical examples in the determination of the PI using PPs. Given previously explained conditions of the BT terms; the use of this theorem should be re-considered. The current determination of the PI using the PPs is not valid for stochastic variables belonging to a stochastic event or process**.

Keywords— **Bayesian statistics; simulation; TCP; NTCP0; product of probabilities.**

I. Introduction

I.1 The derivation of the Bayes' theorem (BT)

Bayesian statistics (BS) is named after Thomas Bayes in 1793 formulated a specific case of Bayes' theorem in a published paper. "Bayes' theorem is a way to figure out conditional probability. Conditional probability is the probability of an event happening, given that it has some relationship to one or more other events." [1]. The BS is widely used in the ionizing radiation field as shown in [2-4]. The Figure 1 shows procedures described in the derivation of the BT.

In the Figure 1, N is the number of people of a population with two stochastic events A and B in These events are characterized with their respective probabilities P(A) and P(B), and

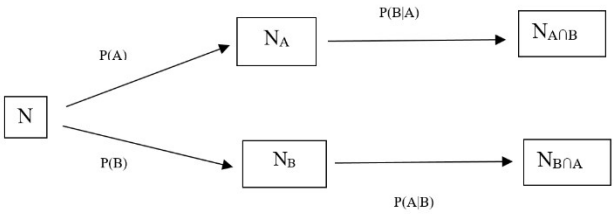

Figure 1. Graphical representation of the probabilistic procedures used in the derivation of the BT of the Eq. (7), and product of probabilities of the Eq. (5) and Eq. (6).

$$N_A = N * P(A) \quad (1)$$
$$N_{A\cap B} = N_A * P(B|A) \quad (2)$$

where N_A is the mean number of people (MNP) with a true A event; while $NA_{\cap B}$ is the MNP with true event A and B, and P(B|A) is the probability of B given A.

$$N_B = N * P(B) \quad (3)$$
$$N_{B\cap A} = N_B * P(A|B) \quad (4)$$

where N_B is the MNP with a true B event; while $N_{B\cap A}$ is the MNP with true both events B and A, where, P(A|B) is the probability of A given B.

Given P(A∩B) is probabilistically defined as $P(A \cap B) = N_{A\cap B}/N$, and using the elements of the Eq. (1) and Eq. (2), then

$$P(A \cap B) = P(A) * P(B|A) \quad (5)$$

The P(B∩A)= P(A∩B), and derived using a similar way is

$$P(B \cap A) = P(B) * \mathrm{P(A|B)} \quad (6)$$

.

Combining the Eq. (5) and Eq. (6), the BT is obtained as the following equation.

$$\mathrm{P(A|B)} = \frac{\mathrm{P(B|A)} * \mathrm{P(A)}}{P(B)} \quad (7)$$

I.2 The product of probabilities (PPs) rule

"This rule states that the probability of simultaneous occurrence of two or more independent events (let's call A and B) is the product of the probabilities of occurrence of each of these events individually" [6], and mathematically expressed as

$$P(A \cap B) = P(A) * P(B) \quad (8)$$

$$P(B \cap A) = P(B) * \mathrm{P(A)} \quad (9)$$

In the radiation oncology therapy, the formulation of the normal tissue complication probability (NTCP) for multiple organs at risk (tNTCP) in [7] used the PPs as

$$tNTCP = 1 - \prod_i (1 - NTCP_i) \quad (10)$$

where $NTCP_i$ is the NTCP for i^{th} organ at risk.

Using the PPs and probabilistic definition, the uncomplicated TCP (UTCP), nowadays, the most acceptable UTCP formulation is the following:

$$UTCP = P(TC \cap NTC0) = TCP * NTCP0 \quad (11)$$

TCP is calculated as the ratio number of patients with a tumor control (TC) and total of a homogenous patient population treated with a specified radiation treatment; and NTCP0 as ratio number of patients without normal tissue complication (NTC0) and total of a homogenous patient population treated with a specified radiation treatment.

II. RESULT AND DISCUSSION

II.1 The Bayes' theorem

Taking into account the Eq. (5) and Eq. (8), as well as Eq. (6) and Eq. (9), then

$$P(A) * P(B|A) = P(A) * P(B) \quad (12)$$

$$P(B) * \mathrm{P(A|B)} = \mathrm{P(B)} * \mathrm{P(A)} \quad (13)$$

The two previous equations show that actually P(B|A) is P(B), and P(A|B) is P(A).

II. 2 Product of probabilities (PPs)

The radiation oncology treatment is a stochastic process (SP) that involves the tumor control (TC), normal tissue non-complication (NTC0) and normal tissue complications, which are independent stochastic variables (SVs), and associated respectively to probabilistic metrics TCP, NTCP0 and $NTCP_i$ (i=1..nc, nc: Number of complications). NTCP0 has been recently introduced in the radiotherapy, as described in [8-10].

When the radiation treatment is given to patients with more than one target, these are treated with more than one treatment. Each separated or successive treatment has its own SVs. For these reasons, TCP and NTCP0 for multiple targets are calculated as

$$TCP = \prod TCP_i \quad (14)$$

$$NTCP0 = \prod NTCP0_i \quad (15)$$

where TCP_i and $NTCP0_i$ are respectively TCP and NTCP0 of the treatment for the i^{th} target. The previous equations can be obtained using the same procedure of the Eq. (5). The Eq. (14) has been previous developed already in [7] by other researchers.

The normal tissue complications associated to $NTCP_i$ or their probabilistic complements ($1\text{-}NTCP_i$) are SVs and simultaneously generated after a radiation treatment; i.e., these processes are not successive, for this reason the PPs of the previous Eq. (10) is incoherent.

In [8] and [10] we have developed a new formulation of the NTCP for multiple organs at risks or End-points, which we have called as the total NTCP (TNTCP=1-NTCP0), and expressed as

$$TNTCP = \sum_i NTCP_i \quad (16)$$

where *i=1..nc, nc:* Number of complications

II. 2 The new P(A∩B) formulation

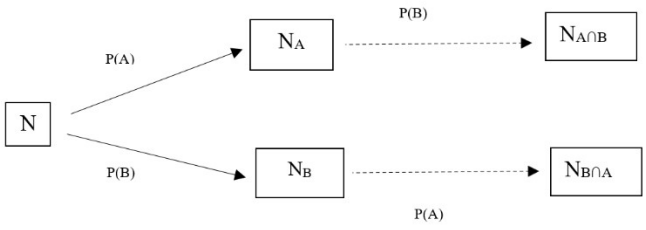

Figure 2. Graphical representation of the incorrect procedures used in the P(A∩B) and P(B∩A) formulations using the Eq. (17) and Eq. (18). Although these equations are probabilistically well-formulated; the dash arrows represent there are not any stochastic process associated.

As shown in the Figure 2 with a dash arow, although the following expressions

$$N_{A \cap B} = N_A * P(B) \quad (17)$$

$$N_{B \cap A} = N_B * P(A) \quad (18)$$

are probabilistically well-formulated; these should be used only for successive SPs, like successive radiation treatments to patients with two targets; and it is incorrect its use when are not associated to SPs. A statistic justification should exist for determining mean values using PPs.

As whatever probabilistic metric, for obtaining P(A∩B) using computational simulations, one should generate one random number (RN) in each simulation; where RN <= P(A∩B) is the condition of success. If a SP has two SVs A and B; given this SP inherently involves a specific P(A) and P(B); in the simulations the generated RN should be compared with P(A) and P(B), in particular with min(P(A),P(B)), which expresses the condition of success for P(A∩B).

The Figure 3 illustrates the procedure used in the derivation of the Eq. (19).

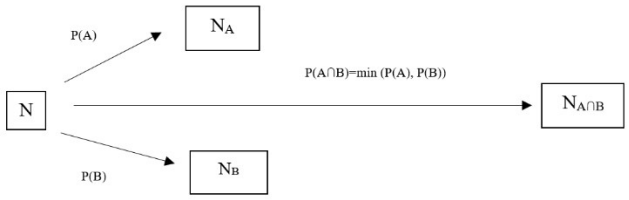

Figure 3. Representation of the two stochastic and independent events or variables A and B with their respective probabilities P(A) and P(B), and our proposal of the joint probability or PI P(A∩B) = min(P(A),P(B)).

$$P(A \cap B) = min(P(A), P(B)) \quad (19)$$

The probabilistic foundation of the Eq. (19) is represented in the Figure 4; and its values are determined for the region of intersection, which is the minimum of the probabilities P(A) and P(B).

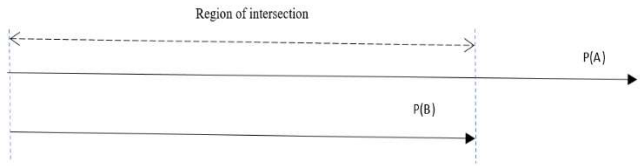

Figure 4. Representation of probabilistic foundation of the PI calculation done by the Eq. (19)

BT example of [1]: You might be interested in finding out a patient's probability of having liver disease if they are an alcoholic. "Being an alcoholic" is the test (kind of like a litmus test) for liver disease.

- A could mean the event "Patient has liver disease." Past data tells you that 10% of patients entering your clinic have liver disease. P(A) = 0.10.
- B could mean the litmus test that "Patient is an alcoholic." Five percent of the clinic's patients are alcoholics. P(B) = 0.05.
- You might also know that among those patients diagnosed with liver disease, are alcoholics. This is your B|A: the probability that a patient is alcoholic, given that they have liver disease, is 7%.
- BT tells you: P(A|B) = (0.07 * 0.1)/0.05 = 0.14. In other words, if the patient is an alcoholic, their chances of having liver disease is 0.14 (14%). This is a large increase from the 10% suggested by past data. But it's still unlikely that any particular patient has liver disease.

The P(B|A) value used in this example (0.07 or 7%) has been assumed.

For determining the probability of patients diagnosed with liver disease, alcoholics too, it should be calculated using the Eq. (19), then its value in this particular case is 5%.

II.3 Application of the P(A∩B)=min(P(A),P(B)) in the radiation oncology therapy.

UTCP is calculated as the ratio of the number of patients with a TC with NTC0 and the total of a homogenous patient population treated with a specified radiation treatment. Although UTCP is related to TCP and NTCP0; like each one of them, UTCP is a probabilistic metric, and this is not a derivation from TCP and NTCP0. Based on the Eq (19) this metric is calculated as

$$UTCP = min(TCP, NTCP0) \quad (20)$$

III) CONCLUSION

For two stochastic variables A and B of a stochastic process, P(A∩B)=min(P(A),P(B)), in other cases P(A∩B)=P(A)*P(B).

As shown in the Eq. (12) and Eq. (13), the P(B|A) and P(A|B) used in the derivation of the BT are not new probabilistic metrics, but they are own respective P(B) and P(A). For this reason, the use of the BT should be re-considered.

The UTCP of the radiation oncology treatments with a known TCP and NTCP0, and calculated as UTCP=min(TCP,NTCP0) is an excellent-practical example in the determination of the PI without using the PPs; but the minimum of the involved probabilities.

The results of UTCP=min(TCP,NTCP0) shows that is more probable the number of patients without normal tissue complications after they being cured with a treatment characterized with a TCP; than of results of

UTCP=TCP*NTCP0 because of min(TCP,NTCP0) >= TCP*NTCP0.

A. Author Contributions Statement

T. Frometa-Castillo wrote the main text; A. Pyakuryal and G. Narayanasamy prepared all figures; and Pyakuryal, Ganesh and Mesbahi reviewed the manuscript.

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'Conflict of interest' statement: None

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